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Qualitative Properties of Solutions of Finite System of Differential Equations Involving R-L Sequential Fractional Derivative

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Abstract: In this paper, qualitative properties such as existence-uniqueness of solutions of finite system of differential equations involving R-L sequential fractional derivative with initial conditions have been studied. Lower and upper solutions are defined for the problem under investigation. Comparison results are used to develop monotone technique for finite system of differential equations involving R-L sequential fractional derivative with initial conditions when the functions on the right hand side are mixed quasi-monotone. Two convergent monotone sequences are obtained by introducing monotone operator. Lipschitz condition is the key part of the study. Minimal and maximal solutions are obtained by using developed technique. Existence and uniqueness of solutions of finite system of differential equations involving R-L sequential fractional derivative is also proved as an application of the technique.

Keywords: Existence and Uniqueness, Sequential Fractional Differential Equations, Lower and Upper Solutions, Monotone Technique

1. Introduction

Fractional differential equations occur more frequently in physics, chemistry, control of dynamical systems etc [2, 3, 6, 7, 19, 21]. During the last two decades many researchers attracted towards existence-uniqueness results for initial value problems (IVPs) [4, 12, 25], boundary value problems (BVPs)[1], periodic boundary value problems (PBVPs) [13, 20] and integral boundary value problems (IBVPS) [22]. Recent results on the theory of differential equations of fractional order due to Lakshmikantham et. al. appeared in [8-11].

Wei et. al. [23, 24] developed monotone iterative scheme for fractional differential equations involving Riemann- Liouville sequential derivative. They have successfully applied the technique to study existence-uniqueness of solution for initial value problems and periodic boundary value problems. Nanware and Dhaigude in 2017 constructed monotone scheme for system of Caputo fractional differential equations with periodic boundary conditions [5, 17], Riemann-Liouville fractional differential equations with integral boundary conditions [15], Riemann-Liouville fractional differential equations with IBCs when the function is quasimonotone non-decreasing [16], Nonlinear System of initial value problems involving sequential Riemann-Liouville fractional derivative [18].

Motivated by above literature, we shall study the following finite system of differential equations involving sequential derivative with initial conditions when the function on the right is mixed quasi-monotone:

$$\begin{aligned}
&(\mathscr{D}^{2\alpha}_{0+}u_i)(p) = f_i(p, u_1, ..., u_N, \mathscr{D}^{\alpha}_{0+}u_1, ..., \mathscr{D}^{\alpha}_{0+}u_N),\\
&p \in (0, T] = J^*\\
&p^{1-\alpha}u_i(p) = u^i_0,\\
&p^{1-\alpha}(\mathscr{D}^{\alpha}u_i)(p)|_{p=0} = u^i_1, \quad i = 1, 2, ..., N.
\end{aligned} \tag{1}$$

where $f_i \in C(J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R}), J = [0, T]$ Monotone technique is developed for the finite system of sequential fractional differential equations (1). Qualitative properties of solutions such as existence-uniqueness are obtained for the problem (1) via monotone technique.

The paper is arranged in the following way:
In the second section definitions and basic results are explored. In third section monotone technique is constructed for finite system of sequential fractional differential equations and technique developed is successfully applied to obtain existence-uniqueness of solution of the finite system of sequential fractional differential equations (1). At the end conclusion is given.

2. Definitions and Basic Results

Definition 2.1. [19] The Riemann-Liouville (R-L) fractional integral of $u(p)$ denoted by $I^{\alpha}_{a+}u(p)$ is

$$(I^{\alpha}_{a+}u)(p) = \frac{1}{\Gamma(\alpha)}\int_a^p (p-s)^{\alpha-1}u(s)ds \qquad 0<\alpha\leq 1$$

and Riemann-Liouville fractional derivative of $u(p)$ denoted by $D^{\alpha}_{a+}u(p)$ is

$$(D^{\alpha}_{a+}u)(p) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dp}\int_a^p (p-s)^{-\alpha}u(s)ds(=\frac{d}{dp}(I^{1-\alpha}_{a+}u)(p)).$$

Definition 2.2. [14] Riemann-Liouville sequential fractional derivative of $u(p)$ denoted by $\mathscr{D}^{\alpha}_{a+}u(p)$ is

$$\mathscr{D}^{\alpha}_{a+}u(p) = D^{\alpha}_{a+}u(p)$$
$$\mathscr{D}^{n\alpha}_{a+}u(p) = \mathscr{D}^{\alpha}_{a+}\mathscr{D}^{(n-1)\alpha}_{a+}u(p), \quad (n=2,3,...).$$

Relation between Riemann-Liouville sequential fractional derivatives and Riemann-Liouville fractional derivatives

In case $k=2, 0<\alpha<\frac{1}{2}$ and R-L derivatives, the relationship between $D^{k\alpha}_{a+}u(p)$ and $\mathscr{D}^{k\alpha}_{a+}u(p)$ is

$$\left(\mathscr{D}^{2\alpha}_{a+}u\right)(p) = \left(D^{2\alpha}_{a+}\left[u(t) - \left(I^{1-\alpha}_{a+}u\right)(a+)\frac{t-a^{\alpha-1}}{\Gamma(\alpha)}\right]\right)(p)$$

Definition 2.3. A function $u_i(p)$ is called a classical solution of finite system of IVP (1) if

1. $u_i(p)$ is continuous on J^*; $p^{1-\alpha}u_i(p)$, $p^{1-\alpha}(\mathscr{D}^{\alpha}u_i)(p)$ are continuous on J, and its fractional integrals $(I^{1-\alpha}u_i)(p)$, $(I^{1-\alpha}D^{\alpha}u_i)(p)$ are continuously differentiable for J^*;
2. $u_i(p)$ satisfies finite system of IVP (1).

Define the following:

$$C([0,T]) = \left\{u_i : \ u_i(p) \text{ is continuous on } J, ||u_i(p)||_C = \max_{p\in J^*}|u_i(p)|\right\}$$
$$C_{1-\alpha}([0,T]) = \left\{u_i \in C(J) : \ p^{1-\alpha}u_i(p)\in C([0,T]), ||u_i(p)||_{C_{1-\alpha}} = ||p^{1-\alpha}u_i(p)||_C\right\}$$
$$C^{\alpha}_{1-\alpha}([0,T]) = \left\{u_i \in C_{1-\alpha}(J) : p^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}u_i)(p)\in C(J)\right\}$$

Definition 2.4. A function $v(p) = (v_1, v_2, ..., v_N) \in C^{\alpha}_{1-\alpha}([0,T])$ is said to be lower solution of finite system of IVP (1) if

$$(\mathscr{D}^{2\alpha}_{0+}v_i)(p) \leq f_i(p, v_1, v_2, ..., v_N, \mathscr{D}^{\alpha}_{0+}v_1, ..., \mathscr{D}^{\alpha}_{0+}v_N), \quad p\in J^*$$
$$p^{1-\alpha}v_i(p)\leq v^i_0, \quad p^{1-\alpha}\left(\mathscr{D}^{\alpha}v_i\right)(p)|_{p=0}\leq v^i_1.$$

A function $w(p) = (w_1, w_2, ..., w_N) \in C^{\alpha}_{1-\alpha}([0,T])$ is said to be upper solution of finite system of IVP (1) if

$$\left(\mathscr{D}^{2\alpha}_{0+}w\right)(p) \geq f_i(p, w_1, w_2, ..., w_N, \mathscr{D}^{\alpha}_{0+}w_1, \mathscr{D}^{\alpha}_{0+}w_2, ..., \mathscr{D}^{\alpha}_{0+}w_N), \quad p\in J^*$$
$$p^{1-\alpha}w_i(p)|_{p=0}\geq w^i_0, \quad p^{1-\alpha}\left(\mathscr{D}^{\alpha}w_i\right)(p)|_{p=0}\geq w^i_0.$$

Definition 2.5. A function $f_i \in C([0,T]\times R^N, R^N)$ is said to satisfy mixed quasimonotone property if for each $i, f_i(t, u_i, [u]_{r_i}, [u]_{s_i})$ is monotone nondecreasing in $[u]_{r_i}$ and monotone nonincreasing in $[u]_{s_i}$.

When either r_i or s_i is equal to zero, mixed quasimonotone property is defined:

Definition 2.6. A function $f_i \in C([0,T]\times R^N, R^N)$ is called quasimonotone nondecreasing (nonincreasing) if for each $i, u_i\leq v_i$ and $u_j = v_j, i\neq j$, then

$$f_i(t, u_1, u_2, ..., u_N)\leq f_i(t, v_1, v_2, ..., v_N) \ \left(f_i(t, u_1, ..., u_N)\geq f_i(t, v_1, ..., v_N)\right).$$

Definition 2.7. Let $f_i(t, u_1, u_2, ..., u_N)$ be real valued continuous function defined on domain $G\subset R^N$.We say that $f_i(t, u_1, u_2, ..., u_N)$ satisfies Lipschitz condition if there exists $L>0$ such that

$$|f_i(t, u_1, u_2, ..., u_N) - f_i(t, \overline{u}_1, \overline{u}_2, ..., \overline{u}_N)|\leq L(|u_i - \overline{u}_i|)$$

for all $(t, u_1, u_2, ..., u_N), \quad (t, \overline{u}_1, \overline{u}_2, ..., \overline{u}_N)\in G$.

Define the sector

$$\Omega = \Big\{(t, u_1, u_2, ..., u_N) \in [0, T] \times R^N : v_i(p) \leq u_i(p) \leq w_i(p), \quad 0 \leq p \leq T\Big\}.$$

Assume that

$$v_i(p) \leq w_i(p), \quad p \in (0, T] : p^{1-\alpha} v_i(p)|_{p=0} \leq p^{1-\alpha} w_i(p)|_{p=0},$$

$$p^{1-\alpha}(\mathscr{D}^{\alpha}_{0+} v_i)(p)|_{p=0} \leq p^{1-\alpha}(\mathscr{D}^{\alpha}_{0+} w_i)(p)|_{p=0}.$$

Define the sector in space $C^{\alpha}_{1-\alpha}([0, T])$:

$$[v, w] = \Big\{u_i \in C^{\alpha}_{1-\alpha}([0, T]) | v_i \leq u_i \leq w_i, p \in (0, T] : p^{1-\alpha} v_i|_{p=0} \leq p^{1-\alpha} u_i|_{p=0}$$

$$\leq p^{1-\alpha} w_i|_{p=0}, t^{1-\alpha}(\mathscr{D}^{\alpha}_{0+} v_i)|_{p=0} \leq p^{1-\alpha}(\mathscr{D}^{\alpha}_{0+} u_i)|_{p=0} \leq p^{1-\alpha}(\mathscr{D}^{\alpha}_{0+} w_i)|_{p=0}\Big\}$$

Following Lemma gives existence of solution for linear initial value problem (LIVP) involving Riemann-Liouville fractional derivative.

Lemma 2.8. Suppose that $u(p) \in C_{1-\alpha}([0, T])$, then the linear initial value problem

$$\mathscr{D}^{\alpha}_{0+} u(p) + M u(p) = \sigma(p), \quad p \in J^*, \quad p^{1-\alpha} u(p)|_{p=0} = u_0, \tag{2}$$

where $M \in \mathbb{R}$ is constant and $\sigma(p) \in C_{1-\alpha}([0, T])$, has following integral representation of solution

$$u(p) = \Gamma(\alpha) u_0 e_{\alpha}(-M, p) + \Big[e_{\alpha}(-M, t) * \sigma(t)\Big](p), \tag{3}$$

where

$$(g * f)(p) = \int_0^p g(p - t) f(t) dt, e_{\alpha}(\lambda, z) = z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^{\alpha}) = z^{\alpha-1} \sum_{l=0}^{\infty} \lambda^l \frac{z^{\alpha l}}{\Gamma[(l+1)\alpha]},$$

where

$$E_{\alpha,\alpha}(p) = \sum_{l=0}^{\infty} \frac{p^k}{\Gamma[(l+1)\alpha]}$$

is Mittag-Leffler function of two parameter.

Lemma 2.9. [18, 23] Suppose that $u(p) \in C^{\alpha}_{1-\alpha}([0, T])$, then linear initial value problem

$$(\mathscr{D}^{2\alpha}_{0+} u)(p) + N \mathscr{D}^{\alpha}_{0+} u(p) + M u(p) = \sigma(p), \quad p \in (0, T],$$

$$x^{1-\alpha} u(x)|_{x=0} = u_0, \quad x^{1-\alpha}(\mathscr{D}^{\alpha}_{0+} u)(x)|_{x=0} = u_1$$

where $N, M \in \mathbb{R}, N^2 \geq 4M$ are constants and $\sigma(p) \in C_{1-\alpha}[0, T]$, has following representation of solution

$$u(p) = \Gamma(\alpha) u_0 e_{\alpha}(\lambda_2, p) + \Gamma(\alpha)(u_1 - \lambda_2 u_0)\Big[e_{\alpha}(\lambda_2, t) * e_{\alpha}(\lambda_1, t)\Big](p) + \Big[e_{\alpha}(\lambda_2, t) * e_{\alpha}(\lambda_1, t) * \sigma(t)\Big](p),$$

where

$$\lambda_1 = \frac{-N + \sqrt{N^2 - 4M}}{2}, \lambda_2 = \frac{-N - \sqrt{N^2 - 4M}}{2} \leq 0.$$

Lemma 2.10. [18, 23]

$$\Big[e_\alpha(\lambda_2,t)*e_\alpha(\lambda_1,t)\Big](p)=\Big[e_\alpha(\lambda_1,t)*e_\alpha(\lambda_2,t)\Big](p)=\frac{1}{\lambda_1-\lambda_2}\Big[e_\alpha(\lambda_1,t)-e_\alpha(\lambda_2,t)\Big](p),\quad p\in\mathbb{R}.$$

Lemma 2.11. [18, 23] For $0\le\alpha\le1$, there exist constants $b_n^0>0,\ b_n^1>0,\ b_n^2>0,...,b_n^n>0$, such that

$$\omega_n(k\alpha)=\sum_{i=0}^{n} b_n^i C_{k+i}^{i+1}.$$

Hence, we have

$$(k-1)\omega_n(k\alpha)=\sum_{l=0}^{n}(l+2)b_n^l C_{k+l}^{l+2}.$$
$$(1+k\alpha)(1+\frac{k\alpha}{2})...(1+\frac{k\alpha}{n})=\frac{1}{\alpha}\sum_{l=0}^{n}\frac{1}{l+1}b_n^l C_{k+l}^{l}.$$

Lemma 2.12. [18, 23] If $0<\alpha\le1$, then $F(p)>0,\ g(p)>0,\ h(p)>0,\forall\ x\in R=(-\infty,+\infty)$, where

$$F(p)=E_{\alpha,\alpha}(p)=\sum_{l=0}^{\infty}\frac{x^l}{\Gamma((l+1)\alpha)},\quad g(p)=\sum_{l=1}^{\infty}\frac{lp^{l-1}}{\Gamma((l+1)\alpha)},\quad h(p)=E_\alpha(p)=\sum_{l=1}^{\infty}\frac{lp^{l-1}}{\Gamma(l\alpha+1)}.$$

Following comparison results play a vital role in the later sections.

Lemma 2.13. [18, 23] If $w\in C_{1-\alpha}([0,T])$ and

$$D^\alpha w(p)+Mw(p)\ge0,\quad p\in J^*$$
$$p^{1-\alpha}w(p)|_{p=0}\ge0,$$

where $M\in\mathbb{R}$ is constant. Then $w(p)\ge0,t\in J^*$.

Lemma 2.14. [18, 23] If $w(p)\in C_{1-\alpha}^{\alpha}([0,T])$ and

$$\mathscr{D}_{0+}^{2\alpha}w(p)+N\mathscr{D}_{0+}^{\alpha}w(p)+Mw(p)=\sigma(p)\ge0,\quad p\in J^*$$
$$p^{1-\alpha}w(p)|_{p=0}=w_0\ge0,\quad p^{1-\alpha}(\mathscr{D}_{0+}^{\alpha}w)(p)|_{p=0}=w_1\ge0$$

where $N,M\in\mathbb{R},\quad N^2\ge4M$ are constants such that

$$\lambda_1=\frac{-N+\sqrt{N^2-4M}}{2}\ge0>\lambda_2=\frac{-N-\sqrt{N^2-4M}}{2}.$$

Then $w(p)\ge0,p\in J^*$.

Suppose that f satisfies the following:

(1) H1: there exist constants $N,M\in\mathbb{R},\quad N^2>4M$ such that

$$f(t,w,\mathscr{D}_{0+}^{\alpha}w)-f(t,v,\mathscr{D}_{0+}^{\alpha}v)\ge-N(\mathscr{D}_{0+}^{\alpha}w-\mathscr{D}_{0+}^{\alpha}v)-M(w-v),$$

$v,w\in C_{1-\alpha}^{\alpha}([0,T])$ are lower-upper solutions of the finite system of initial value problem (1)

(2) H2: there exist constants $N,M\in\mathbb{R},\quad N^2>4M$ such that $(H1)$ holds and for $p\in(0,T],\quad v(p)\le y_2\le y_1\le w(p),\quad D_1(p)\le z_i\le D_2(p),\quad i=1,2$ such that

$$f(p,y_1,z_1)-f(p,y_2,z_2)\ge-N(z_1-z_2)-M(y_1-y_2),\tag{4}$$

where

$$D_1(p)=(\mathscr{D}_{0+}^{\alpha}v)(p)+\lambda_2(w(p)-v(p)),$$
$$D_2(p)=(\mathscr{D}_{0+}^{\alpha}w)(p)-\lambda_2(w(p)-v(p)),\quad p\in(0,T],$$

$$\lambda_1 = \frac{-N + \sqrt{N^2 - 4M}}{2} \geq 0 > \lambda_2 = \frac{-N - \sqrt{N^2 - 4M}}{2}. \tag{5}$$

(3) H3: there exist constants $N, M \in \mathbb{R}, \quad N^2 > 4M$ such that $(H2)$ holds and for $p \in (0,T], \quad v(p) \leq y_2 \leq y_1 \leq w(p), \quad D_1(p) \leq z_i \leq D_2(p), i = 1, 2$ such that

$$f(p, y_1, z_1) - f(p, y_2, z_2) \leq N(z_1 - z_2) + M(y_1 - y_2),$$

In view of (4),

$$f(t, u, v) + Mu + Nv$$

is monotone non-decreasing in u, v for $u, v \in C_{1-\alpha}([0,T])$.

Lemma 2.15. [18, 23] If $(H1)$ holds then

$$\mathscr{D}^{\alpha}_{0+}(w - v)(p) - \lambda_2(w - v)(p) \geq 0, \quad p \in J^*.$$

Hence,

$$\mathscr{D}^{\alpha}_{0+}(w)(p) - \lambda_2(w(p) - v(p)) \geq \mathscr{D}^{\alpha}_{0+}v)(p) \geq \mathscr{D}^{\alpha}_{0+}v)(p) + \lambda_2(w(p) - v(p)), p \in J^*,$$

where $\lambda_2 < 0$ is given by (5).

Lemma 2.16. [18, 23] If $(H1)$ holds then

$$\Omega = \{\eta \in [v, w] : D_1(p) \leq (\mathscr{D}^{\alpha}_{0+}\eta)(p) \leq D_2(p), \ p \in J^*\}$$

is a convex closed set.

3. Monotone Technique

Monotone technique is developed in the section for finite system of sequential fractional differential equations with Riemann-Liouville fractional derivative and developed method is successfully implemented to study existence-uniqueness of solutions of finite system of sequential fractional differential equations (1).

Theorem 3.1. Assume that

(1) $v_0^i, w_0^i \in C^{\alpha}_{1-\alpha}([0,T])$ are ordered lower-upper solutions finite system of sequential fractional differential equations (IVP) (1) and $f_i \in C([0,T] \times \mathbb{R}^{\mathbb{N}})$,

(2) f_i satisfies Lipschitz condition (one-sided)

$$f_i(p, w_1, ..., w_N, \mathscr{D}^{\alpha}_{0+}w_1, ..., \mathscr{D}^{\alpha}_{0+}w_n) - f_i(p, v_1, ..., v_N, \mathscr{D}^{\alpha}_{0+}v_1, ..., \mathscr{D}^{\alpha}_{0+}v_N) \geq -N_i(\mathscr{D}^{\alpha}_{0+}w_i - \mathscr{D}^{\alpha}_{0+}v_i) - M_i(w_i - v_i),$$

where $N_i, M_i \in \mathbb{R}, \quad N_i^2 > 4M_i$.

(3) There exist constants $N_i, M_i \in \mathbb{R}, \quad N_i^2 > 4M_i$ such that (ii) holds and for $p \in (0,T], v_i \leq \beta_i \leq \alpha_i \leq w_i, \quad D_1^i \leq z_j^i \leq D_2^i, j = 1, 2,$ such that

$$f_i(p, \alpha_1, \alpha_2, ..., \alpha_N, z_1^1, z_2^1) - f_i(p, \beta_1, \beta_2, ..., \beta_N, z_1^2, z_2^2) \leq N_i(z_i^1 - z_i^2) + M_i(\alpha_i - \beta i)$$

where

$$D_1^i = \mathscr{D}^{\alpha}_{0+}v_i(p) + \lambda_2^i(w_i(t) - v_i(p)), D_2^i = \mathscr{D}^{\alpha}_{0+}w_i(p) - \lambda_2^i(w_i(p) - v_i(p))$$

$$\lambda_1^i = \frac{-N_i + \sqrt{N_i^2 - 4M_i}}{2} \geq 0 > \lambda_2^i = \frac{-N_i - \sqrt{N_i^2 - 4M_i}}{2}$$

Then there exist sequences $\{v_{in}(p)\}, \{w_{in}(p)\} \subset C^{\alpha}_{1-\alpha}([0,T])$ with $v_0^i = v_i, w_0^i = w_i$ such that for $p \in (0,T]$

$$\lim_{n \to \infty} v_{in}(p) = \rho_i(p), \quad \lim_{n \to \infty} w_{in}(p) = \gamma_i(p)$$

and ρ_i, γ_i are minimal and maximal solutions on $[v, w]$ for system of IVP respectively and for any solution u_i of system of IVP such that $u_i(p) \in \Omega$, we have

$$v_0^i \leq v_{i1} \leq v_{i2} \leq ... \leq v_{in} \leq \rho_i \leq u_i \leq \gamma_i \leq ... \leq w_{i2} \leq w_{i1} \leq w_0^i.$$

Proof. Let

$$\sigma(\eta_i)(p) = f_i(p, \eta_1, \eta_2, ..., \eta_N \mathscr{D}^{\alpha}_{0+}\eta_1, \mathscr{D}^{\alpha}_{0+}\eta_2, ..., \mathscr{D}^{\alpha}_{0+}\eta_N) + N_i \mathscr{D}^{\alpha}_{0+}\eta_i(p) + M_i\eta_i(p), \quad p \in (0, T].$$

For any $\eta = (\eta_1, \eta_2, ..., \eta_N) \in \Omega$, consider the linear initial value problem (LIVP)

$$\begin{aligned}
&(\mathscr{D}^{2\alpha}_{0+}u_i)(p) + N_i\mathscr{D}^{\alpha}_{0+}u_i(p) + M_i u_i(p) = \sigma(\eta_i)(p), \quad p \in (0, T],\\
&p^{1-\alpha}u_i(p)|_{p=0} = u^i_0, \quad p^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}u_i)(p)|_{p=0} = u^i_1.
\end{aligned}$$

By Lemma 2.8 and relation

$$\left(\mathscr{D}^{2\alpha}_{a+}u\right)(p) = \left(D^{2\alpha}_{a+}\left[u(p) - \left(I^{1-\alpha}_{a+}u\right)(a+)\frac{p - a^{\alpha-1}}{\Gamma(\alpha)}\right]\right)(x)$$

LIVP has exactly one solution $u_i(x) \in C^{\alpha}_{1-\alpha}([0, T])$ and is given by

$$u_i(p) = (A\eta_i)(p) = \Gamma(\alpha)u^i_0 e_\alpha(\lambda^i_2, p) + \Gamma(\alpha)(u^i_1 - \lambda_2 u^i_0)\Big[e_\alpha(\lambda^i_2, p) * e_\alpha(\lambda^i_1, p)\Big](p) + \Big[e_\alpha(\lambda^i_2, p) * e_\alpha(\lambda^i_1, p) * \sigma\eta_i(p)\Big](p),$$

where

$$\lambda^i_1 = \frac{-N_i + \sqrt{N^2_i - 4M_i}}{2}, \quad \lambda^i_2 = \frac{-N_i - \sqrt{N^2_i - 4M_i}}{2} \le 0. \tag{6}$$

$$\begin{aligned}
(\mathscr{D}^{\alpha}_{0+}A\eta_i)(p) &= \Gamma(\alpha)u^i_0\lambda^i_2 e_\alpha(\lambda^i_2, p) + \Gamma(\alpha)(u^i_1 - \lambda_2 u^i_0)\frac{1}{\lambda^i_1 - \lambda^i_2}\Big[\lambda^i_1 e_\alpha(\lambda^i_1, p) - \lambda^i_2 e_\alpha(\lambda^i_1, p)\Big](p)\\
&+ \frac{1}{\lambda^i_1 - \lambda^i_2}\Big\{\lambda^i_1 e_\alpha(\lambda^i_1, t) * \sigma(\eta_i)(t) - \lambda^i_2 e_\alpha(\lambda^i_2, p) * (\sigma\eta_i)(p)\Big\}(p).
\end{aligned}$$

Then A is an operator from Ω into $C^{\alpha}_{1-\alpha}([0, T])$ and $\eta_i(p)$ is a solution of finite system of FDE if and only if $\eta_i = A\eta_i$. Since $\lambda^i_1 \ge 0 \ge \lambda^i_2$ in (6), we have

$$\frac{1}{\lambda^i_1 - \lambda^i_2}\Big[\lambda^i_1 e_\alpha(\lambda^i_1, p) - \lambda^i_2 e_\alpha(\lambda^i_2, p)\Big](p) \ge 0, \quad p \in (0, T].$$

Using Lemma 2.10 and Lemma 2.11, lower and upper solutions, definition of $u_i(t)$ and $(\mathscr{D}^{\alpha}_{0+}A\eta_i)(x)$, we get

$$\begin{aligned}
v_i(p) = (A\eta_i)(p) &= \Gamma(\alpha)u^i_0 e_\alpha(\lambda^i_2, p) + \Gamma(\alpha)(u^i_1 - \lambda^i_2 u^i_0)\Big[e_\alpha(\lambda^i_2, t) * e_\alpha(\lambda^i_1,)\Big](p) + \Big[e_\alpha(\lambda^i_2, p) * e_\alpha(\lambda^i_1, p) * (\sigma\eta_i)(p)\Big](p)\\
v_i(p) \le Av_i(p) &= \Gamma(\alpha)u^i_0 e_\alpha(\lambda^i_2, p) + \Gamma(\alpha)(u^i_1 - \lambda^i_2 u^i_0)\Big[e_\alpha(\lambda^i_2, p) * e_\alpha(\lambda^i_1, p)\Big](p) + \Big[e_\alpha(\lambda^i_2, p) * e_\alpha(\lambda^i_1, p) * \sigma v_i(p)\Big](p)\\
&\le (A\eta_i)(p) \le (Aw_i)(p) \le w_i(p), \quad \forall \quad \eta_i(p) \in \Omega.
\end{aligned}$$

That is

$$v_i(p) \le Av_i(p) \le A\eta_i(p) \le Aw_i(p) \le w_i(p), \quad \forall \quad \eta_i(p) \in \Omega \tag{7}$$

and if

$$\begin{aligned}
v_i(p) \le \theta_i(p) \le \phi_i(p) \le w_i(p) \ \text{ then } (\sigma\theta_i)(p) \le (\sigma\phi_i)(p), A\theta_i(p) \le A\phi_i(p),\\
\text{and } \mathscr{D}^{\alpha}_{0+}A\theta_i(p) \le \mathscr{D}^{\alpha}_{0+}A\phi_i(p).
\end{aligned} \tag{8}$$

By Lemma 2.8, for $i = 1, 2$, we have

$$z^i_1(p) = \mathscr{D}^{\alpha}_{0+}(A\eta_i - v_i)(p) - \lambda^i_2(A\eta_i - v_i)(p) \ge 0, \ t \in (0, T], \forall \ \eta_i(p) \in \Omega.$$

Hence

$$\mathscr{D}^{\alpha}_{0+}(A\eta_i)(p) = \mathscr{D}^{\alpha}_{0+}v_i(p) + \lambda^i_2(A\eta_i - v_i)(p) \ge \mathscr{D}^{\alpha}_{0+}v_i(p) + \lambda^i_2(w_i - v_i)(p) = D^i_1(p)$$

Similarly, we can show

$$\mathscr{D}^{\alpha}_{0+}(A\eta_i)(p) - \mathscr{D}^{\alpha}_{0+}v_i(p) + \lambda_2^i(A\eta_i - v_i)(p) \geq 0, \quad p \in (0,T], \quad \forall \quad \eta_i(p) \in \Omega.$$
$$\therefore \mathscr{D}^{\alpha}_{0+}(A\eta_i)(p) \geq \mathscr{D}^{\alpha}_{0+}v_i(p) - \lambda_2^i(A\eta_i - v_i)(p) \geq \mathscr{D}^{\alpha}_{0+}v_i(p) - \lambda_2^i(w_i - v_i)(p) = D_2^i(p).$$

Therefore

$$A(\Omega) \subset \Omega.$$

Now let

$$v_0^i = v_i, \quad w_0^i = w_i, \quad v_{in} = Av_{i,n-1}, \quad w_{in} = Aw_{i,n-1}, n = 1,2,3,...$$

From (7) and (8), we have

$$v_i(p) \leq v_{i1}(p) \leq v_{i2}(p) \leq ... \leq v_{in}(p) \leq ... \leq w_{in}(x) \leq ... \leq w_{i2}(p) \leq w_{i1}(p) \leq w_i(p)$$

$$D_1^i(p) \leq \mathscr{D}^{\alpha}_{0+}v_{i1}(p) \leq \mathscr{D}^{\alpha}_{0+}v_{i2}(p) \leq ... \leq \mathscr{D}^{\alpha}_{0+}v_{in}(p) \leq ... \leq \mathscr{D}^{\alpha}_{0+}w_{in}(p) \leq ... \leq \mathscr{D}^{\alpha}_{0+}w_{i2}(p) \leq \mathscr{D}^{\alpha}_{0+}w_{i1} \leq D_2^i(p)$$

Clearly, the upper sequence w_{ik} is monotone non-decreasing and bounded below and that lower sequence v_{ik} is monotone non-decreasing and bounded above. Moreover, $\mathscr{D}^{\alpha}_{0+}v_{ik},\ \mathscr{D}^{\alpha}_{0+}w_{ik} \in [D_1^i(p), D_2^i(p)]$.
Let $B_i = \{v_{in} : n = 1,2,3,...\}$. Now we show that the set B_i is relatively compact in $C^{\alpha}_{1-\alpha}([0,T])$. For any $\eta_i(p) \in \Omega$, by definition of lower and upper solutions and Lipschitz condition, we have

$$\begin{aligned}(\mathscr{D}^{2\alpha}_{0+}v_i)(p) + N_i\mathscr{D}^{\alpha}_{0+}v_i(p) + M_iv_i(p) &\leq f_i(x, v_1, v_2, ..., v_N\mathscr{D}^{\alpha}_{0+}v_1, \mathscr{D}^{\alpha}_{0+}v_2, ..., \mathscr{D}^{\alpha}_{0+}v_n) + N_i\mathscr{D}^{\alpha}_{0+}v_i(p) + M_iv_i(p)\\ &\leq f_i(p, \eta_1, \eta_2, ..., \eta_N\mathscr{D}^{\alpha}_{0+}\eta_1, \mathscr{D}^{\alpha}_{0+}\eta_2, \mathscr{D}^{\alpha}_{0+}\eta_N) + N_i\mathscr{D}^{\alpha}_{0+}\eta_i(p) + M_i\eta_i(p)\\ &\leq f_i(p, w_1, w_2, ..., w_n\mathscr{D}^{\alpha}_{0+}w_1, \mathscr{D}^{\alpha}_{0+}w_2, ..., \mathscr{D}^{\alpha}_{0+}w_n) + N_i\mathscr{D}^{\alpha}_{0+}w_i(p) + M_iw_i(p)\end{aligned}$$

Since $B_i,\ \Omega \subset C^{\alpha}_{1-\alpha}([0,T])$ are bounded sets, therefore

$$\{\sigma\eta_i(p) = f_i(x, \eta_1, \eta_2, \mathscr{D}^{\alpha}_{0+}\eta_1, \mathscr{D}^{\alpha}_{0+}\eta_2) + N_i\mathscr{D}^{\alpha}_{0+}\eta_i(p) + M_i\eta_i(p) | \eta_i \in \Omega\}$$

is bounded. Thus there exists constant $L > 0$ such that

$$\|\sigma(v_{ik})(p)\| = max|p^{1-\alpha}\sigma(v_{ik}(p)| \leq L,\ \forall, \quad k = 1,2,.... \Leftrightarrow |\sigma(v_{ik}(p)| \leq Lp^{1-\alpha}, \forall \quad p \in (0,T].$$

On the other hand by using Lemma 2.8, $\{v_k(p) | k \in N\}$ satisfies

$$\begin{aligned}v_{ik}(p) &= \Gamma(\alpha)y_0^ie_\alpha(\lambda_2^i, p) + \Gamma(\alpha)(y_1^i - \lambda_2^iy_0^i)\Big[e_\alpha(\lambda_2^i, p) * e_\alpha(\lambda_1^i, p)\Big](p)\\ &+ \Big[e_\alpha(\lambda_2^i, p) * e_\alpha(\lambda_1^i, p) * (\sigma(v_{i,k-1})(p)\Big](p)\end{aligned} \tag{9}$$

$$\begin{aligned}(\mathscr{D}^{\alpha}_{0+}v_{ik})(p) &= \Gamma(\alpha)y_0^i\lambda_2e_\alpha(\lambda_2^i, p) + \Gamma(\alpha)(y_1^i - \lambda_2^iy_0^i)\frac{1}{\lambda_1^i - \lambda_2^i}\Big[\lambda_1^ie_\alpha(\lambda_1^i, p) - \lambda_2^ie_\alpha(\lambda_2^i, p)\Big](p)\\ &+ \frac{1}{\lambda_1^i - \lambda_2^i}\Big[\lambda_1^ie_\alpha(\lambda_1^i, p) * \sigma(v_{i,k-1})(p) - \lambda_2^ie_\alpha(\lambda_2^i, p) * \sigma(v_{i,k-1})(p)\Big](p)\end{aligned} \tag{10}$$

Let $G(\lambda_j^i, p) = p^{1-\alpha}\Big[e_\alpha(\lambda_j^i, p) * \sigma(v_{i,k-1})(p)\Big], p \in [0,T], j = 1,2$. Without loss of generality, we assume that $0 \leq t_1 < t_2 \leq T$, from $\lambda_2^i < 0 \leq \lambda_1^i$, we have

$$\begin{aligned}\Big|G(\lambda_2^i, t_1) - G(\lambda_2^i, t_2)\Big| &= \Big|t_1^{1-\alpha}\Big(e_\alpha(\lambda_2^i, t) * \sigma(v_{i,k-1})(t_1)\Big) - t_2^{1-\alpha}\Big(e_\alpha(\lambda_2^i, t) * \sigma(v_{i,k-1})(t_2)\Big)\Big|\\ &= \Big|t_1^{1-\alpha}\int_0^x e_\alpha(\lambda_2^i, x - t_1)\sigma(v_{i,k-1})(t_1)dt - t_2^{1-\alpha}\int_0^x e_\alpha(\lambda_2^i, t)\sigma(v_{i,k-1})(t_2)dt\Big|\\ &\leq \frac{L\Gamma(\alpha)}{|\lambda_1^i|}\Big|E_{\alpha,\alpha}(\lambda_2^i, t_1^\alpha) - E_{\alpha,\alpha}(\lambda_2^i, t_2^\alpha)\Big| + \frac{2L\Gamma(\alpha)}{\Gamma(2\alpha)}(t_2 - t_1)^\alpha\end{aligned} \tag{11}$$

and

$$\begin{aligned}\Big|G(\lambda_1^i,t_1)-G(\lambda_1^i,t_2)\Big| &= \Big|t_1^{1-\alpha}[e_\alpha(\lambda_1^i,t)*\sigma(v_{i,k-1})(t_1)] = \Big|t_1^{1-\alpha}\int_0^x e_\alpha(\lambda_1^i,x-t_1)\sigma(v_{i,k-1})(t_1)dt \\ &- t_2^{1-\alpha}\int_0^x e_\alpha(\lambda_1^i,t)\sigma(v_{i,k-1})(t_2)dt\Big| \le \left(\frac{L\Gamma(\alpha)}{|\lambda_1^i|}+\frac{LT^\alpha}{\alpha}\right)\Big|E_{\alpha,\alpha}(\lambda_1^i,t_1^\alpha)-E_{\alpha,\alpha}(\lambda_1^i,t_2^\alpha)\Big| \\ &+ \frac{2L\Gamma(\alpha)}{\Gamma(2\alpha)}E_{\alpha,\alpha}(\lambda_1^i,T^\alpha(t_2-t_1)^\alpha)\end{aligned} \tag{12}$$

From $E_{\alpha,\alpha}(p)\in C([0,T]),\quad \forall \quad \epsilon>0$, there exists $\delta=\delta(\epsilon)$, when $|t_1-t_2|<\delta$ (without loss of generality $0\le t_1<t_2\le T$), we have

$$\Big|E_{\alpha,\alpha}(\lambda_1^i,t_1^\alpha)-E_{\alpha,\alpha}(\lambda_1^i,t_2^\alpha)\Big| < \frac{\epsilon}{8L_1} \tag{13}$$

$$\Big|E_{\alpha,\alpha}(\lambda_2^i,t_1^\alpha)-E_{\alpha,\alpha}(\lambda_2^i,t_2^\alpha)\Big| < \frac{\epsilon}{8L_2} \tag{14}$$

$$(t_2-t_1)^\alpha < \frac{\epsilon}{8L_3} \tag{15}$$

where

$$\begin{aligned}L_1 &= max\left\{\frac{|\Gamma(\alpha)(y_1^i-\lambda_2^i y_0^i)\lambda_1^i|}{|\lambda_1^i-\lambda_2^i|},\frac{1}{|\lambda_1^i-\lambda_2^i|}\left(\Gamma(\alpha)+\frac{|\lambda_1^i|T^\alpha}{\alpha}\right)\right\} \\ L_2 &= max\left|\Gamma(\alpha)(y_0^i\lambda_2^i),\frac{|\Gamma(\alpha)(y_1^i-\lambda_2^i y_0^i)\lambda_1^i|}{|\lambda_1-\lambda_2|},\frac{L\Gamma(\alpha)}{|\lambda_1-\lambda_2|}\right| \\ L_3 &= \frac{2L\Gamma(\alpha)}{\Gamma(2\alpha)|\lambda_1^i-\lambda_2^i|}\Big(|\lambda_2^i|+|\lambda_1^i|\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda_1^i|T^\alpha)\Big)\end{aligned}$$

Using (11) to (13) in (10), we obtain

$$\begin{aligned}&\Big|t_1^{1-\alpha}(\mathscr{D}_{0+}^\alpha v_{ik})(t_1)-t_2^{1-\alpha}(\mathscr{D}_{0+}^\alpha v_{ik})(t_2)\Big| = \Big|t_1^{1-\alpha}\Gamma(\alpha)u_0^i e_\alpha(\lambda_2^i,t_1)+t_1^{1-\alpha}\Gamma(\alpha)(u_1^i-\lambda_2^i u_0^i) \\ &- \frac{1}{\lambda_1^i-\lambda_2^i}[\lambda_1^i e_\alpha(\lambda_1^i,t)-\lambda_2^i e_\alpha(\lambda_2^i,t)](t_1)+t_1^{1-\alpha}\Big[\lambda_1^i e_\alpha(\lambda_1^i,t)*\sigma(v_{i,k-1})(t)-\lambda_2^i e_\alpha(\lambda_2^i,t)*\sigma(v_{i,k-1})(t)\Big](t_1) \\ &- t_2^{1-\alpha}\Gamma(\alpha)u_0^i e_\alpha(\lambda_2^i,t)-t_2^{1-\alpha}\Gamma(\alpha)(u_1^i-\lambda_2^i u_0^i)\frac{1}{\lambda_1^i-\lambda_2^i}\Big[\lambda_1^i e_\alpha(\lambda_1^i,t)-\lambda_2^i e_\alpha(\lambda_2^i,t)](t_2)\Big] \\ &- t_2^{1-\alpha}\Big[\lambda_1^i e_\alpha(\lambda_1^i,t)*\sigma(v_{i,k-1})(t)-\lambda_2^i e_\alpha(\lambda_2^i,t)*\sigma(v_{i,k-1})(t)\Big](t_2)\Big| \le \Big|\Gamma(\alpha)u_0^i\lambda_2^i\Big|.\Big|E_{\alpha,\alpha}(\lambda_2^i t_1^\alpha)-E_{\alpha,\alpha}(\lambda_2^i t_2^\alpha)\Big| \\ &+ \frac{|\Gamma(\alpha)(u_1^i-\lambda_2^i u_0^i)}{|\lambda_1^i-\lambda_2^i|}\Big\{|\lambda_1^i||E_{\alpha,\alpha}(\lambda_1^i t_1^\alpha)-E_{\alpha,\alpha}(\lambda_1^i t_2^\alpha)|+|\lambda_2^i||E_{\alpha,\alpha}(\lambda_2^i t_1^\alpha)-E_{\alpha,\alpha}(\lambda_2^i t_2^\alpha)|\Big\} \\ &+ \frac{L}{|\lambda_1^i-\lambda_2^i|}\left((\Gamma(\alpha)+\frac{|\lambda|T^\alpha}{\alpha})|E_{\alpha,\alpha}(\lambda_1^i t_1^\alpha)-E_{\alpha,\alpha}(\lambda_1^i t_2^\alpha)|+\Gamma(\alpha)|E_{\alpha,\alpha}(\lambda_2^i t_1^\alpha)-E_{\alpha,\alpha}(\lambda_2^i t_2^\alpha)|\right) \\ &+ \frac{2L\Gamma(\alpha)}{\Gamma(2\alpha)|\lambda_1^i-\lambda_2^i|}\Big(|\lambda_2^i|+|\lambda_1^i|\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda_1^i|T^\alpha)\Big)(t_2-t_1)^\alpha \le \epsilon.\end{aligned}$$

Thus B_i is equicontinuous in $C_{1-\alpha}^\alpha([0,T])$, by Ascoli-Arzela theorem, we have that B_i is relatively compact set of $C_{1-\alpha}^\alpha([0,T])$. Similarly we can prove $\{w_{ik}(p)\}$ is relatively compact set of $C_{1-\alpha}^\alpha([0,T])$. Hence, the sequences $\{v_{ik}(p)\},\{w_{ik}(p)\}$ converges uniformly to $\rho_i(p),\gamma_i(p)$ respectively on $[0,T]$ i.e.

$$\lim_{k\to\infty}v_{ik}(p)=\rho_i(p)\quad,\lim_{k\to\infty}w_{ik}(x)=\gamma_i(x),\quad p\in[0,T]$$
$$\lim_{k\to\infty}\mathscr{D}_{0+}^{2\alpha}v_{ik}(p)=\mathscr{D}_{0+}^{2\alpha}\rho_i(p)\quad,\lim_{k\to\infty}\mathscr{D}_{0+}^{2\alpha}w_{ik}(p)=\mathscr{D}_{0+}^{2\alpha}\gamma_i(p),\quad p\in[0,T].$$

Thus by relations $(v_i \leq v_{i1}...)$, it follows that v_i and w_i satisfy

$$\begin{aligned} v_i \leq v_{i1} \leq v_{i2} \leq ... \leq v_{in} \leq \rho_i \leq \gamma_i \leq ... \leq w_{in} \leq ... \leq w_{i2} \leq w_{i1} \leq w_i \\ D_1^i(p) \leq \mathscr{D}_{0+}^{\alpha} v_{i1} \leq \mathscr{D}_{0+}^{\alpha} v_{i2} \leq \mathscr{D}_{0+}^{\alpha} \rho_i \leq \mathscr{D}_{0+}^{\alpha} \gamma_i \leq ... \leq \mathscr{D}_{0+}^{\alpha} v_{in} \leq ... \\ \leq \mathscr{D}_{0+}^{\alpha} w_{in} \leq ... \leq \mathscr{D}_{0+}^{\alpha} w_{i2} \leq \mathscr{D}_{0+}^{\alpha} w_{i1} \leq D_2^i(p) \end{aligned} \tag{16}$$

Lastly we prove $\rho_i(p), \gamma_i(p)$ are minimal and maximal solutions of finite system of IVP . Since f_i is continuous , $\sigma(\eta_i)(p)$ is continuous and is monotone non-decreasing in v_i, the sequence v_{ik} converges to $\rho_i(p)$ implies that $\sigma(v_{ik})(p)$ converges to $\sigma(\rho_i)(x),\ x \in (0,T]$.

Taking limit as $k \to \infty$ of v_{ik} i.e equation (4.4) and by dominated convergence theorem, $\rho_i(p)$ satisfies the integral equation

$$\begin{aligned} \rho_i(p) = (A\rho_i)(p) = \Gamma(\alpha) y_0^i e_\alpha(\lambda_2^i, p) + \Gamma(\alpha)(y_1^i - \lambda_2^i y_0^i)\Big[e_\alpha(\lambda_2^i, t) * e_\alpha(\lambda_1^i, t)\Big](p) \\ + \Big[e_\alpha(\lambda_2^i, p) * e_\alpha(\lambda_1^i, p) * (\sigma\rho_i)(p)\Big](p) \end{aligned}$$

Thus $\rho_i(p)$ is an integral representation of the solution to LIVP, that is $\rho_i(p)$ is an integral representation of the solution of finite system of IVP.

Since f_i is continuous and by Lemma 2.1, $\rho_i(p)$ is a classical solution of finite system of IVP. This proves that lower sequence $v_{ik}(p)$ converges to a solution $\rho_i(p)$ of finite system of IVP. Similarly we can show that the upper sequence $w_{ik}(p)$ converges to a solution $\gamma_i(p)$ of finite system of IVP and satisfies $\rho_i(p) \leq \gamma(i)(p), \mathscr{D}_{0+}^{\alpha}\rho_i(p) \leq \mathscr{D}_{0+}^{\alpha}\gamma_i(p),\ p \in (0,T]$. Thus by standard arguments it follows that

$$v_i \leq v_{i1} \leq v_{i2} \leq ... \leq v_{in} \leq \rho_i \leq \gamma_i \leq ... \leq w_{in} \leq ... \leq w_{i2} \leq w_{i1} \leq w_i$$

and hence $\rho_i(p)$ and $\gamma_i(p)$ are minimal-maximal solutions of finite system of IVP on $[v,w]$ respectively.

Finally if for $p \in (0,T], v_i \leq y_2^i \leq y_1^i \leq w_i, D_1^i(p) \leq z_j^i \leq D_2^i(p), j = 1,2$ there exists N_i, M_i such that

$$f_i(p, y_1, y_2, z_1, z_2) - f_i(p, \overline{y_1}, \overline{y_2}, \overline{z_1}, \overline{z_2}) \leq N_i(z_i - \overline{z_i}) + M_i(y_i - \overline{y_i})$$

$f_i(p, y_1, y_2, z_1, z_2)$ is quasimonotone nondecreasing in y_1, y_2, Z_1, Z_2
for $y, z \in C_{1-\alpha}([0,T]))$ then $\rho_i(p) = \gamma_i(p)$ is a unique solution of finite system of IVP. It is sufficient to prove $\rho_i(p) \leq \gamma_i(p), \mathscr{D}_{0+}^{\alpha}\rho_i(p) \geq \mathscr{D}_{0+}^{\alpha}\gamma_i(p),\ p \in (0,T]$. Thus finite system of IVP and above hypothesis gives, for $w_i(p) = \rho_i(p) - \gamma_i(p)$

$$\begin{aligned} (\mathscr{D}_{0+}^{2\alpha} w_i)(p) + N_i \mathscr{D}_{0+}^{\alpha} w_i(p) + M_i w_i(p) &= \mathscr{D}_{0+}^{2\alpha}(\rho_i - \gamma_i)(p) + N_i \mathscr{D}_{0+}^{\alpha}(\rho_i - \gamma_i)(p) + M_i(\rho_i - \gamma_i)(p) \\ &= \mathscr{D}_{0+}^{2\alpha}\rho_i - \mathscr{D}_{0+}^{2\alpha}\gamma_i)(p) + N_i \mathscr{D}_{0+}^{\alpha}\rho_i(p) - N_i \mathscr{D}_{0+}^{\alpha}\gamma_i(p) + M_i\rho_i(p) - M_i\gamma_i(p) = (\phi w_i)(p) \geq 0, \quad p \in (0,T] \end{aligned}$$

$$x^{1-\alpha} w_i(p)|_{p=0} = 0, \quad p^{1-\alpha}(\mathscr{D}_{0+}^{\alpha} w_i)(p)|_{p=0} = 0.$$

$$\mathscr{D}_{0+}^{\alpha} w_i)(p) = \frac{1}{\lambda_1^i - \lambda_2^i}\left\{\left(\lambda_1^i e_\alpha(\lambda_1^i, p) - \lambda_2^i e_\alpha(\lambda_2^i, p)\right) * (\phi w_i)(p)\right\}(p)$$

Then by Lemma 2.11, we have $w_i(p) = 0, \quad p \in (0,T]$. Thus

$$\rho_i(p) = \gamma_i(p), \quad \mathscr{D}_{0+}^{\alpha}\rho_i(p) = \mathscr{D}_{0+}^{\alpha}\gamma_i(p), \quad p \in (0,T].$$

Therefore we obtain $\rho_i(p) = u_i(p) = \gamma_i(p)$ is a solution of finite system of initial value problem (1).

Theorem 3.2. Assume that

(1) $v_0^i, w_0^i \in C_{1-\alpha}^{\alpha}([0,T])$ are ordered lower- upper solutions of the finite system of initial value problem IVP (1) and $f_i \in C([0,T] \times \mathbb{R}^{\mathbb{N}})$,

(2) f_i satisfies Lipschitz condition

$$|\Delta| \leq M_i|v_i - w_i| + N_i|\mathscr{D}_{0+}^{\alpha} v_i - \mathscr{D}_{0+}^{\alpha} w_i|, \tag{17}$$

where

$$\begin{aligned}\Delta =& f_i(p, x_1, ..., x_N, \mathscr{D}^{\alpha}_{0+}x_1, ..., \mathscr{D}^{\alpha}_{0+}x_N) - f_i(p, y_1, ..., y_N, \mathscr{D}^{\alpha}_{0+}y_1, ..., \mathscr{D}^{\alpha}_{0+}y_N);\\ & x_i, y_i \in [v, w], \quad \mathscr{D}^{\alpha}_{0+}x_i, \mathscr{D}^{\alpha}_{0+}y_i \in [D^i_1(p), D^i_2(p)],\\ \text{and} \quad & M_i > 0, N_i > 0, \quad N_i^2 > 4M_i\end{aligned}$$

are Lipschitz constant such that

$$\begin{aligned}&D^i_1 = \mathscr{D}^{\alpha}_{0+}x_i(p) + \lambda^i_2(y_i(p) - x_i(p)), D^i_2 = \mathscr{D}^{\alpha}_{0+}y_i(p) - \lambda^i_2(y_i(p) - x_i(p)),\\ &\lambda^i_1 = \frac{-N_i + \sqrt{N_i^2 - 4M_i}}{2} \geq 0 > \lambda^i_2 = \frac{-N_i - \sqrt{N_i^2 - 4M_i}}{2}\end{aligned}$$

Then finite system of IVP (1) has unique solution in sector $[v, w]$.
Proof. From (17), we have

$$-M_i(x_i - y_i) - N_i(\mathscr{D}^{\alpha}_{0+}x_i - \mathscr{D}^{\alpha}_{0+}y_i) \leq \Delta \leq M_i(x_i - y_i) + N_i(\mathscr{D}^{\alpha}_{0+}x_i - \mathscr{D}^{\alpha}_{0+}y_i)$$

where

$$\begin{aligned}\Delta =& f_i(p, x_1, ..., x_N, \mathscr{D}^{\alpha}_{0+}x_1, ..., \mathscr{D}^{\alpha}_{0+}x_N) - f_i(p, y_1, ..., y_N, \mathscr{D}^{\alpha}_{0+}y_1, ..., \mathscr{D}^{\alpha}_{0+}y_N)\\ & v_i \leq x_i \leq y_i \leq w_i, \quad \mathscr{D}^{\alpha}_{0+}x_i, \mathscr{D}^{\alpha}_{0+}y_i \in [D^i_1(p), D^i_2(p)].\end{aligned}$$

By Theorem 3.1, the finite system of IVP (1) has unique solution in the sector $[v, w]$.

4. Conclusion

We have developed monotone technique for finite system of sequential fractional differential equations with initial conditions when the function on the right side is mixed quasi monotone using lower and upper solutions. It is successfully applied to prove qualitative properties such as existence and uniqueness of solutions of the problem under investigation.

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